

Poynting vector and poynting theorem

When electromagnetic wave travels in space, it carries energy and energy density is always associated with electric fields and magnetic fields.

The rate of energy travelled through per unit area i.e. the amount of energy flowing through per unit area in the perpendicular direction to the incident energy per unit time is called poynting vector.

Mathematically poynting vector is represented as

$$\vec{P} = \vec{E} \times \vec{H} \left(= \frac{\vec{E} \times \vec{B}}{\mu} \right) \quad \dots(i)$$

the direction of poynting vector is perpendicular to the plane containing \vec{E} and \vec{H} . Poynting vector is also called as instantaneous energy flux density. Here rate of energy transfer \vec{P} is perpendicular to both \vec{E} and \vec{H} . Since it represents the rate of energy transfer per unit area, its unit is W/m².

Poynting theorem states that the net power flowing out of a given volume V is equal to the time rate of decrease of stored electromagnetic energy in that volume decreased by the conduction losses.

i.e. total power leaving the volume = rate of decrease of stored electromagnetic energy
– ohmic power dissipated due to motion of charge

Proof : The energy density carried by the electromagnetic wave can be calculated using Maxwell's equations

$$\text{as } \text{div } \vec{D} = 0 \quad \dots(i) \quad \text{div } \vec{B} = 0 \quad \dots(ii) \quad \text{Curl } \vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad \dots(iii)$$

$$\text{and } \text{Curl } \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t} \quad \dots(iv)$$

taking scalar product of (iii) with H and (iv) with \vec{E}

$$\text{i.e. } \vec{H} \text{ curl } \vec{E} = -\vec{H} \cdot \frac{\partial \vec{B}}{\partial t} \quad \dots(v)$$

$$\text{and } \vec{E} \cdot \text{curl } \vec{H} = \vec{E} \cdot \vec{J} + \vec{E} \cdot \frac{\partial \vec{D}}{\partial t} \quad \dots(vi)$$

doing (vi) – (v) i.e.

$$\begin{aligned}\vec{H} \cdot \text{curl } \vec{E} - \vec{E} \cdot \text{curl } \vec{H} &= -\vec{H} \cdot \frac{\partial \vec{B}}{\partial t} - \vec{E} \cdot \vec{J} - \vec{E} \cdot \frac{\partial \vec{D}}{\partial t} \\ &= -\left[\vec{H} \cdot \frac{\partial \vec{B}}{\partial t} + \vec{E} \cdot \frac{\partial \vec{D}}{\partial t} \right] - \vec{E} \cdot \vec{J}\end{aligned}$$

as $\text{div}(\vec{A} \times \vec{B}) = \vec{B} \cdot \text{curl } \vec{A} - \vec{A} \cdot \text{curl } \vec{B}$

so $\text{div}(\vec{E} \times \vec{H}) = -\left[\vec{H} \cdot \frac{\partial \vec{B}}{\partial t} + \vec{E} \cdot \frac{\partial \vec{D}}{\partial t} \right] - \vec{E} \cdot \vec{J}$... (vii)

But $\vec{B} = \mu \vec{H}$ and $\vec{D} = \epsilon \vec{E}$

so
$$\begin{aligned}\vec{H} \cdot \frac{\partial \vec{B}}{\partial t} &= \vec{H} \cdot \frac{\partial (\mu \vec{H})}{\partial t} = \frac{1}{2} \mu \frac{\partial (\vec{H} \cdot \vec{H})}{\partial t} \\ &= \frac{\partial}{\partial t} \left[\frac{1}{2} \vec{H} \cdot \vec{B} \right]\end{aligned}$$

and
$$\vec{E} \cdot \frac{\partial \vec{D}}{\partial t} = \vec{E} \cdot \frac{\partial (\epsilon \vec{E})}{\partial t} = \frac{1}{2} \epsilon \frac{\partial (\vec{E} \cdot \vec{E})}{\partial t} = \frac{\partial}{\partial t} \left[\frac{1}{2} \vec{E} \cdot \vec{D} \right]$$

so from equation (vii) $\text{div}(\vec{E} \times \vec{H}) = -\frac{\partial}{\partial t} \left[\frac{1}{2} (\vec{H} \cdot \vec{B} + \vec{E} \cdot \vec{D}) \right] - \vec{E} \cdot \vec{J}$

or
$$\vec{E} \cdot \vec{J} = -\frac{\partial}{\partial t} \left[\frac{1}{2} (\vec{H} \cdot \vec{B} + \vec{E} \cdot \vec{D}) \right] - \text{div}(\vec{E} \times \vec{H})$$
 ... (viii)

Integrating equation (viii) over a volume V enclosed by a surface S

$$\int_V \vec{E} \cdot \vec{J} dV = -\int_V \frac{\partial}{\partial t} \left\{ \frac{1}{2} (\vec{H} \cdot \vec{B} + \vec{E} \cdot \vec{D}) \right\} dV - \int_V \text{div}(\vec{E} \times \vec{H}) dV$$

(24)

$$\text{or} \quad \int_V \vec{E} \cdot \vec{J} \, dV = - \int_V \left[\frac{1}{2} \mu H^2 + \frac{1}{2} \epsilon E^2 \right] dV - \int_S (\vec{E} \times \vec{H}) \cdot d\vec{s}$$

$$\text{as} \quad \vec{B} = \mu \vec{H}, \quad \vec{D} = \epsilon \vec{E} \quad \text{and} \quad \int_V \text{div}(\vec{E} \times \vec{H}) dV = \int_S (\vec{E} \times \vec{H}) \cdot d\vec{s}$$

$$\text{or} \quad \int_V (\vec{E} \cdot \vec{J}) dV = - \frac{\partial}{\partial t} \int_V \left[\frac{1}{2} \mu H^2 + \frac{1}{2} \epsilon E^2 \right] dV - \int_S (\vec{E} \times \vec{H}) \cdot d\vec{s}$$

$$\text{or} \quad \int_S (\vec{E} \times \vec{H}) \cdot d\vec{s} = - \int_V \frac{\partial U_{em}}{\partial t} dV - \int_V (\vec{E} \cdot \vec{J}) dV$$

$$\text{or} \quad \boxed{\int_S \vec{P} \cdot d\vec{s} = - \int_V \frac{\partial U_{em}}{\partial t} dV - \int_V (\vec{E} \cdot \vec{J}) dV} \quad (\text{as } \vec{P} = \vec{E} \times \vec{H}) \quad \dots(\text{ix})$$

i.e. Total power leaving the volume = rate of decrease of stored e.m. energy -
ohmic power dissipated due to charge motion

This equation (ix) represents the poynting theorem according to which the net power flowing out of a given volume is equal to the rate of decrease of stored electromagnetic energy in that volume minus the conduction losses.

In equation (ix) $\int_S \vec{P} \cdot d\vec{s}$ represents the amount of electromagnetic energy crossing the closed surface per second or the rate of flow of outward energy through the surface S enclosing volume V i.e. it is poynting vector.

The term $\int_V \frac{\partial U_{em}}{\partial t} dV$ or $\frac{\partial}{\partial t} \int_V \left[\frac{1}{2} \mu H^2 + \frac{1}{2} \epsilon E^2 \right] dV$, the terms $\frac{1}{2} \mu H^2$ and $\frac{1}{2} \epsilon E^2$ represent

the energy stored in electric and magnetic fields respectively and their sum denotes the total energy stored in electromagnetic field. So total terms gives the rate of decrease of energy stored in volume V due to electric and magnetic fields.

$\int_V (\vec{E} \cdot \vec{J}) dV$ gives the rate of energy transferred into the electromagnetic field.

This is also known as work-energy theorem. This is also called as the energy conservation law in electromagnetism.